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Hawking Radiation Arising from the Electromagnetic Fields in the Kerr–Newman Black Hole

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Hawking radiation arising from the electromagnetic fields in the Kerr–Newman black hole is studied exactly by using the Newman–Penrose formalism and the tortoise coordinate. It is shown that the thermal radiation spectrum due to the photons in the Kerr–Newman black hole does not depend on the spins of the particles, and the effect is exactly same as that of the Klein–Gordon scalar particles.

Since Hawking's original discovery (Hawking, 1975) of black hole thermal radiance by using quantum field theory on a given classical background spacetime, quantum thermal radiation of the black holes has been studied extensively. The thermal effect of the Klein–Gordon scalar field in the Kerr–Newman black hole is studied by Damour and Ruffini (1976), the NUT–Kerr–Newman black hole by Ahmed (1987), the Vaidya–Schwarzschild–de-Sitter by Dai *et al.* (1993), the Vaidya–Bouner by Dai and Zhao (1992), and the radiating, rotating charged black hole by Jing and Wang (1996, 1997). The quantum thermal radiation of the Dirac field in the Kerr black hole was investigated by Liu and Xu (1980), the Kerr–Newman black hole by Zhao and Gui (1983), Kerr–Newman–Kasuya by Ahmed and Mondal (1993). However, Liu and Xu (1980) and Zhao and Gui (1983) found that the thermal radiation spectrum for the Dirac particles is same as that of the Klein–Gordon particles only for the near extreme black holes.

To best of our knowledge we know the thermal effect of the electromagnetic field in the black hole spacetime has not been studied yet. Whether or not the thermal effect of the electromagnetic field is same as that of Klein–Gordon field is an interesting open question. The purpose of this paper is to investigate the quantum thermal effect of the photons in the Kerr–Newman black hole spacetime.

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In this paper we first express the Maxwell equations of the Kerr–Newman black hole in Newman–Penrose formalism. We then seek decoupled electromagnetic equations, and after that we investigated quantum thermal effect of the photons by introducing the tortoise coordinate.

The metric of the Kerr–Newman black hole (Kerr, 1963; Newman *et al.*, 1965) in Boyer–Lindquist coordinates is described by

$$g_{\mu\nu} = \begin{pmatrix} -\dfrac{\Delta_r - a^2\sin^2\theta}{\Sigma} & 0 & 0 & -\dfrac{a\sin^2\theta(r^2+a^2-\Delta_r)}{\Sigma} \\ 0 & \dfrac{\Sigma}{\Delta_r} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ -\dfrac{a\sin^2\theta(r^2+a^2-\Delta_r)}{\Sigma} & 0 & 0 & \left(\dfrac{(r^2+a^2)^2-\Delta_r a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta \end{pmatrix}, \tag{1}$$

with

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta_r = (r - r_+)(r - r_-), \tag{2}$$

where $r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2}$, and r_+, M, Q, and a represent the radius of the event horizon, the mass, the charge, and the angular momentum per unit mass of the black hole, respectively.

In order to express the Maxwell equations in the spacetime (1) in the Newman–Penrose formalism, we choose the components of the null tetrad vectors as

$$\begin{aligned} l^\mu &= \frac{1}{\Delta_r}((r^2+a^2), \Delta_r, 0, \ a), \\ n^\mu &= \frac{1}{2\Sigma}((r^2+a^2), -\Delta_r, 0, \ a), \\ m^\mu &= -\frac{\bar{\rho}}{\sqrt{2}}\left(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\right), \\ \bar{m}^\mu &= -\frac{\rho}{\sqrt{2}}\left(-ia\sin\theta, 0, 1, -\frac{i}{\sin\theta}\right). \end{aligned} \tag{3}$$

The corresponding covariant quantities can be taken as

$$\begin{aligned} l_\mu &= \frac{1}{\Delta_r}(\Delta_r, -\Sigma, 0, -a\Delta_r\sin^2\theta), \\ n_\mu &= \frac{1}{2\Sigma}(\Delta_r, \Sigma, 0, -a\Delta_r\sin^2\theta), \end{aligned}$$

$$m_\mu = -\frac{\bar{\rho}}{\sqrt{2}}(ia\sin\theta, 0, -\Sigma, -i(r^2+a^2)\sin\theta),$$
$$\bar{m}_\mu = -\frac{\rho}{\sqrt{2}}(-ia\sin\theta, 0, -\Sigma, i(r^2+a^2)\sin\theta), \tag{4}$$

and the nonvanishing spin-coefficients are (Carmeli, 1982; Chandrasekhar, 1992)

$$\rho = -\frac{1}{r - ia\cos\theta}, \qquad \beta = -\frac{\bar{\rho}\cot\theta}{2\sqrt{2}},$$
$$\pi = \frac{ia\rho^2\sin\theta}{\sqrt{2}}, \qquad \tau = -\frac{ia\rho\bar{\rho}\sin\theta}{\sqrt{2}},$$
$$\mu = -\frac{\rho^2\bar{\rho}\Delta_r}{2}, \qquad \gamma = \mu + \frac{\rho\bar{\rho}(r-M)}{2}, \qquad \alpha = \pi - \bar{\beta}, \tag{5}$$

The tree independent complex tetrad components of the Maxwell tensors $F_{\mu\nu}$ are given by (Carmeli, 1982)

$$\phi_0 = F_{\mu\nu}l^\mu m^\nu,$$
$$\phi_1 = \frac{1}{2}F_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu),$$
$$\phi_2 = F_{\mu\nu}\bar{m}^\mu n^\nu, \tag{6}$$

Thus we get the Maxwell equations for the electromagnetic field in tetrad notation (Carmeli, 1982)

$$D\phi_1 - \bar{\delta}\phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2 + 2\pi J_1,$$
$$\delta\phi_1 - \Delta\phi_0 = (\mu - 2\gamma)\phi_0 - 2\tau\phi_1 - \sigma\phi_2 + 2\pi J_3,$$
$$D\phi_2 - \delta\phi_1 = -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\epsilon)\phi_2 + 2\pi J_4,$$
$$\delta\phi_2 - \Delta\phi_1 = -\nu\phi_0 + 2\mu\phi_1 + (\tau - 2\beta)\phi_2 + 2\pi J_2, \tag{7}$$

where the intrinsic derivatives $D = l^\mu\partial_\mu$, $\Delta = n^\mu\partial_\mu$, $\delta = m^\mu\partial_\mu$, $\bar{\delta} = \bar{m}^\mu\partial_\mu$, and the tetrad components J_n of the four current j_μ are given by $J_n = j_\mu Z_n^\mu$. For the source free electromagnetic field in the Kerr–Newman black hole spacetime we have $J_n = 0$.

By using the Maxwell equation (7) and the following commutation relation (Carmeli, 1982)

$$[D - (p+1)\epsilon + \bar{\epsilon} + q\rho - \bar{\rho}](\delta - p\beta + q\tau)$$
$$- [\delta - (p+1)\beta - \bar{\alpha} + \bar{\pi} + q\tau](D - p\epsilon + q\rho) = 0, \tag{8}$$

we obtain the decoupled equations

$$[(D-\epsilon+\bar{\epsilon}-2\rho-\bar{\rho})(\Delta+\mu-2\gamma)$$
$$-(\delta-\beta-\bar{\alpha}-2\tau+\bar{\pi})(\bar{\delta}+\pi-2\alpha)]\phi_0=0, \tag{9}$$

$$[(\Delta+\gamma-\bar{\gamma}+2\mu+\bar{\mu})(D-\rho+2\epsilon)$$
$$-(\bar{\delta}+\alpha+\bar{\beta}+2\pi-\bar{\tau})(\delta-\tau+2\beta)]\phi_2=0, \tag{10}$$

Substituting the intrinsic derivatives and the spin-coefficients (5) into the decoupled equations (10), and letting

$$\Phi_0=\phi_0=R_{+1}(r)\Theta_{+1}(\theta)\,e^{-i(\omega t-m\varphi)},$$
$$\Phi_2=\frac{2\phi_2}{\bar{\rho}^2}=R_{-1}(r)\Theta_{-1}(\theta)\,e^{-i(\omega t-m\varphi)}, \tag{11}$$

we obtain the separated equations

$$(\Delta_r\mathcal{D}_1\mathcal{D}_1^\dagger-2iEr)R_{+1}(r)=\lambda R_{+1}(r),$$
$$(\Delta_r\mathcal{D}_0^\dagger\mathcal{D}_0+2iEr)R_{-1}(r)=\lambda R_{-1}(r),$$
$$(\mathcal{L}_0^\dagger\mathcal{L}_1+2aE\cos\theta)\Theta_{+1}(\theta)=-\lambda\Theta_{+1}(\theta),$$
$$(\mathcal{L}_0\mathcal{L}_1^\dagger+2aE\cos\theta)\Theta_{-1}(\theta)=-\lambda\Theta_{-1}(\theta), \tag{12}$$

with

$$\mathcal{D}_n=\frac{\partial}{\partial r}+\frac{iK_1}{\Delta_r}+2n\frac{r-M}{\Delta_r},$$
$$\mathcal{D}_n^\dagger=\frac{\partial}{\partial r}-\frac{iK_1}{\Delta_r}+2n\frac{r-M}{\Delta_r},$$
$$\mathcal{L}_n=\frac{\partial}{\partial\theta}+K_2+n\cot\theta,$$
$$\mathcal{L}_n^\dagger=\frac{\partial}{\partial\theta}-K_2+n\cot\theta. \tag{13}$$

where $K_1=(r^2+a^2)\omega-ma$ and $K_2=a\omega\sin\theta-\frac{m}{\sin\theta}$. The Eq. (12) can be explicitly expressed as

$$\Delta_r\frac{d^2R_s}{dr^2}+4(r-M)\frac{dR_s}{dr}$$
$$+\left[2s+4isrE+\frac{K_1^2}{\Delta_r}-\frac{2isK_1(r-M)}{\Delta_r}-\lambda\right]R_s=0,\qquad (s=+1),$$

$$\Delta_r \frac{d^2 R_s}{dr^2} + \left[+4isrE + \frac{K_1^2}{\Delta_r} - \frac{2isK_1(r-M)}{\Delta_r} - \lambda \right] R_s = 0, \quad (s=-1),$$
$$\frac{d^2\Theta_s}{d\theta^2} + \cot\theta \frac{d\Theta_s}{d\theta} + \left[2maE - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} \right.$$
$$\left. + 2asE\cos\theta + \frac{2sm\cos\theta}{\sin^2\theta} - s - s^2\cot^2\theta + \lambda \right]\Theta_s = 0, \quad (s=+1),$$
$$\frac{d^2\Theta_s}{d\theta^2} + \cot\theta \frac{d\Theta_s}{d\theta} + \left[2maE - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} \right.$$
$$\left. + 2asE\cos\theta + \frac{2sm\cos\theta}{\sin^2\theta} + s - s^2\cot^2\theta + \lambda \right]\Theta_s = 0, \quad (s=-1),$$
$$(14)$$

We now introduce the tortoise coordinate in the Kerr–Newman black hole as

$$r_* = r + \frac{1}{2\kappa}\ln(r - r_+) \tag{15}$$

with

$$\kappa = \frac{r_+ - r_-}{2(r_+^2 + a^2)}, \tag{16}$$

where κ is the surface gravity of the black hole. Then we have

$$\frac{d}{dr} = \left[1 + \frac{1}{2\kappa(r - r_+)}\right]\frac{d}{dr_*},$$
$$\frac{d^2}{dr^2} = \left[1 + \frac{1}{2\kappa(r - r_+)}\right]^2 \frac{d^2}{dr_*^2} - \frac{1}{2\kappa(r-r_+)^2}\frac{d}{dr_*^2}, \tag{17}$$

Substituting the new coordinate into Eq. (14) we know the radial quantities near the event horizon can be expressed as

$$\frac{d^2 R_s}{dr_*^2} + 2s\kappa\frac{dR_s}{dr_*} + [(\omega - \Omega_H m)^2 - 2is\kappa(\omega - \Omega_H m)]R_s = 0, \quad (s = \pm 1), \tag{18}$$

where $\Omega_H = \frac{a}{r_+^2 + a^2}$ is the angular velocity of the black hole. Solving Eq. (18) exactly we find that both the radial functions for the $s = \pm 1$ particles are given by

$$R_s(r_*) = N_\omega e^{\pm i k_{r_*} r_*},$$
$$k_{r_*} = (\omega - \Omega_H m), \quad (\text{for} \quad s = \pm 1), \tag{19}$$

where N_ω is an arbitrary integral constant. We know from above discussion that the two linearly independent radial solutions for the photons can be written as

$$\Phi^{\rm in}(v,\hat r) = N_{\rm in}\, e^{-i\omega v},$$
$$\Phi^{\rm out}(v,\hat r) = N_{\rm out}\, e^{-i\omega v}\, e^{2i\omega\hat r}, \tag{20}$$

where

$$v = t + \hat r = t + \frac{1}{\omega}(\omega - \Omega_H m) r_* \tag{21}$$

is an advance Eddington–Finkelstein coordinate, $\Phi^{\rm in}(v,\hat r)$ represents an incoming wave and is an analytic function on the event horizon; $\Phi^{\rm out}(v,\hat r)$, however, represents an outgoing wave and has a logarithmic singularity on the horizon. Near the event horizon, noting the coordinate $\hat r$ tends to

$$\hat r \sim \frac{1}{2\kappa}\ln(r - r_+), \tag{22}$$

we have

$$\Phi^{\rm out}(v,\hat r) = N_{\rm out}\, e^{-i\omega v}(r - r_+)^{i\omega/\kappa}. \tag{23}$$

We now extend the outgoing wave outside the horizon to the region inside. Since on the horizon the outgoing wave function is not analytic we cannot be extended straightforwardly to the region inside. It must be continued analytically in the complex plane by going around the event horizon. Thus inside the horizon, we get

$$\Phi^{\rm out}(v,\hat r) = N_{\rm out}\, e^{-i\omega v}(r_+ - r)^{i\omega/\kappa}\, e^{\pi\omega/\kappa}. \tag{24}$$

We can generally rewrite the outgoing wave function for both inside and outside region as

$$\Phi^{\rm out}(v,\hat r) = N_{\rm out}\{y(r - r_+)\, e^{-i\omega v}(r - r_+)^{i\omega/\kappa} + y(r_+ - r)\, e^{-i\omega v}(r_+ - r)^{i\omega/\kappa}\, e^{\pi\omega/\kappa}\}, \tag{25}$$

where $y(x)$ is a step function

$$y(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases} \tag{26}$$

According to the suggestion of Damour and Ruffini (1976) and Sannan (1988), and by using the normalization condition, we have

$$N_{\rm out}^2 = \frac{\Gamma_\omega}{\exp(2\pi\omega/\kappa) - 1} = \frac{\Gamma_\omega}{\exp[2\pi\omega/(\kappa_{\rm B} T)] - 1}, \tag{27}$$

with

$$T = \frac{\kappa}{2\pi\kappa_{\rm B}}, \tag{28}$$

where Γ_ω is the transmission coefficient caused by the potential barrier in the exterior gravitational field, k_B the Boltzmann constant. The formula (27) is the main result demonstrating the emission of a thermal spectrum of the electromagnetic field in the Kerr–Newman black hole. The temperature of the thermal radiation is given by (28).

To summary, we first express the Maxwell equations in the Kerr–Newman black hole as the Newman–Penrose formalism and obtain the decouple equations. Then we study the Hawking radiation of the Kerr–Newman black hole by solving the electromagnetic field equations exactly in region near the event horizon with the tortoise coordinate. We find that the quantum thermal radiation spectrum due to the electromagnetic field in the Kerr–Newman black hole does not depend on the spins of the particles, and the effect is exactly same as that arising from the Klein–Gordon scalar field. The result is also valid for the electromagnetic fields in the Kerr black hole, the Reissner–Nordström black hole, and the Schwarzschild black hole.

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